A game-theoretic version of Oakes' example for randomized forecasting

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Abstract

Using the game-theoretic framework for probability, Vovk and Shafer [10] have shown that it is always possible, using randomization, to make sequential probability forecasts that pass any countable set of well-behaved statistical tests. This result generalizes work by other authors, who consider only tests of calibration.

We complement this result with a lower bound. We show that Vovk and Shafer's result is valid only when the forecasts are computed with unrestrictedly increasing degree of accuracy.

When some level of discreteness is fixed, we present a game-theoretic generalization of Oakes' example for randomized forecasting that is a test failing any given method of deferministic forecasting; originally, this example was presented for deterministic calibration.

Key words: Universal prediction, Randomized prediction, Randomized rounding, Calibration, Game-theoretic approach to probability, Oakes' example

1. Introduction

Using the game-theoretic framework for probability [9], Vovk and Shafer have shown in [10] that it is always possible, using randomization, to make sequential probability forecasts that pass any countable set of well-behaved statistical tests. This result generalizes work by other authors, among them are Foster and Vohra [4], Kakade and Foster [5], Lehrer [6], Sandrony et al. [8], who consider only tests of calibration.

We complement this result with a lower bound. We show that Vovk and Shafer's result is valid only when the forecasts are computed with unrestrictedly increasing degree of accuracy. When some level of discreteness is fixed, we present a game-theoretic version

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of Oakes' example for randomized forecasting that is a test failing any given method of deterministic forecasting; originally, this example was presented for deterministic calibration. To formulate this example, we use the forecasting game presented by Vovk and Shafer [10], namely Binary Forecasting Game II.

We discuss details of the randomized forecasting algorithms in Section 2.

The Shafer and Vovk's [9] game-theoretic framework is considered in Section 3. We present in this section the original Vovk and Shafer's [10] result on universal randomized forecasting and prove our result which gives the limits for such forecasting - a game-theoretic version of the Oakes' example for randomized forecasting.

2. Background

The research discussed in this paper was started from a notion of calibration, originated by Dawid [1,2]. A test of calibration checks whether the observed empirical frequencies of state occurrences converge to their forecaster probabilities.

Let $I(p)$ denote the indicator function of a subinterval $I \subseteq [0,1]$, i.e., $I(p) = 1$ if $p \in I$, and $I(p) = 0$, otherwise. An infinite sequence of forecasts $p_1, p_2, \dots$ is *well-calibrated* for an infinite sequence of outcomes $\omega_1\omega_2\dots$ if for the characteristic function $I(p)$ of any subinterval of $[0,1]$ *the calibration error* tends to zero, i.e.,

$$\frac{\sum_{i=1}^{n} I(p_i)(\omega_i - p_i)}{\sum_{i=1}^{n} I(p_i)} \to 0 \tag{1}$$

as the denominator of the relation (1) tends to infinity. Here, $I(p_i)$ determines some "selection rule" which defines indices i where we compute the deviation between forecasts p_i and outcomes ω_i.

The main problem of sequential forecasting is to define a universal forecasting algorithm which computes forecasts p_n given past observations $\omega_1, \dots, \omega_{n-1}$ for each n. This universal prediction algorithm should be well-calibrated for each infinite sequence of outcomes. Oakes [7] proposed arguments (see Dawid [3] for a different proof) that no such algorithm can be well-calibrated for all possible sequences: any forecasting algorithm cannot be calibrated for the sequence $\omega = \omega_1\omega_2\dots$, where

$$\omega_i = \begin{cases} 1 \text{ if } p_i < 0.5 \\ 0 \text{ otherwise} \end{cases}$$

and p_i are forecasts computed by the algorithm given $\omega_1, \dots, \omega_{i-1}$, $i = 1, 2, \dots$. The corresponding intervals are $I_0 = [0, 0.5)$ and $I_1 = [0.5, 1]$. It is easy to see that the condition (1) of calibration fails for this ω, where $I = I_0$ or $I = I_1$.

Foster and Vohra [4] show that calibration is almost surely guaranteed with a randomizing forecasting rule, i.e., where the forecasts are chosen using internal randomization. Kakade and Foster [5] noticed that some calibration results require very little randomization. They defined "an almost deterministic" *randomized rounding* universal forecasting algorithm f: for any sequence of outcomes $\omega_1\omega_2\dots$, an observer can only randomly round the deterministic forecast up to Δ in order to calibrate with the internal probability 1

$$\left| \frac{1}{n} \sum_{i=1}^{n} I(p_i)(\omega_i - p_i) \right| \le \Delta, \tag{2}$$

where Δ is the calibration error, $I(p)$ is the indicator function of an arbitrary subinterval of $[0, 1]$.

This approach was further developed by, among others, Lehrer [6], Sandrony et al. [8]. These papers were only concerned with asymptotic calibration. Non-asymptotic version of randomized forecasting was proposed by Vovk and Shafer [10] and by Vovk et al. [11]. They based on the game-theoretic framework of Shafer and Vovk [9].

Let $\mathcal{P}\{0, 1\}$ be the set of all measures on the two-element set $\{0, 1\}$. Any measure from $\mathcal{P}\{0, 1\}$ is represented by a number $p \in [0, 1]$ - the probability of $\{1\}$. We consider also the set of all measures on $\mathcal{P}\{0, 1\}$. Let $\mathcal{P}[0, 1]$ be the set of all probability measures on the unit interval $[0, 1]$ supplied with the standard Borel σ-field $\mathcal{F}$.

Randomizing forecasting is defined as follows. For each n, a forecaster given a binary sequence of past outcomes $\omega_1 \dots \omega_{n-1}$ (and a sequence of past forecasts $p_1, \dots, p_{n-1}$) outputs a probability distribution $P_n \in \mathcal{P}[0, 1]$. The forecasts p_n of the the future event $\omega_n = 1$ are distributed according to this probability distribution.

Assume for each n, the probability distribution P_n is concentrated on a finite subset D_n of $[0, 1]$, say, $D_n = \{p_{n,1}, \dots, p_{n,m_n}\}$. The number $\Delta = \liminf_{n\to\infty} \Delta_n$, where

$$\Delta_n = \inf\{|p_{n,i} - p_{n,j}| : i \ne j\},$$

is called *the level of discreteness* of the corresponding forecasting scheme on the sequence $\omega_1\omega_2 \dots$.

In general case D_n is a predictable random variable, i.e., measurable with respect to the σ-field $\mathcal{F}^{n-1}$, depending on $\omega_1 \dots \omega_{n-1}$.

A typical example is the uniform rounding: for each n, rational points $p_{n,i}$ divide the unit interval into equal parts of size $0 < \Delta < 1$ and P_n is concentrated on these points. In this case the level of discreteness on arbitrary sequence $\omega_1\omega_2 \dots$ equals Δ.

3. Game-theoretic randomized forecasting

Shafer and Vovk [9] proposed a game-theoretic framework for probability theory. In Vovk and Shafer [10] they used this framework to demonstrate the possibility of good probability forecasting in a general setting. This result generalizes the previous work of many authors.

Vovk and Shafer presented a game between Reality, Forecaster and Skeptic. In this game, Forecaster faces Skeptic whose strategy is revealed in advance, and he is allowed to use a degree of randomization to conceal each of his probability forecasts until the corresponding outcome has been announced. Their main result says that Forecaster can keep Skeptic from becoming infinitely reach. Intuitively, this means that the outcomes determined by Reality look random with respect to probability forecasts. This result is a consequence of the von Neumann's minimax theorem.

Vovk and Shafer [10] consider a perfect-information game of randomized forecasting - *Binary Forecasting Game II* between three players - Forecaster, Skeptic, Reality, Random Number Generator described by the following *protocol*:

Let $\mathcal{K}_0 = 1$ and $\mathcal{F}_0 = 1$.
FOR $n = 1, 2, \ldots$
Skeptic announces $S_n : [0,1] \to \mathcal{R}$.
Forecaster announces a probability distribution $P_n \in \mathcal{P}[0,1]$.
Reality announces $\omega_n \in \{0,1\}$.
Forecaster announces $f_n : [0,1] \to \mathcal{R}$ such that $\int f_n(p)P_n(dp) \leq 0$.
Random Number Generator announces $p_n \in [0,1]$.
Skeptic updates his capital $\mathcal{K}_n = \mathcal{K}_{n-1} + S_n(p_n)(\omega_n - p_n)$.
Forecaster updates his capital $\mathcal{F}_n = \mathcal{F}_{n-1} + f_n(p_n)$.
ENDFOR

Restriction on Skeptic: Skeptic must choose the S_n so that his capital $\mathcal{K}_n$ is nonnegative for all n no matter how the other players move.

Restriction on Forecaster: Forecaster must choose the P_n and f_n so that his capital $\mathcal{F}_n$ is nonnegative for all n no matter how the other players move.

Vovk and Shafer [10] showed that Forecaster has a winning strategy in the Forecasting Game II, where Forecaster wins if either (i) his capital $\mathcal{F}_n$ is unbounded or (ii) Skeptic's capital $\mathcal{K}_n$ stays bounded; otherwise the other players win.

Theorem 1 *Forecaster has a winning strategy in Binary Forecasting Game II.*

Sketch of the proof. For completeness of the presentation, we reproduce the proof from [10]. The proof is based on von Neumann's minimax theorem.

At first, at any round n of Binary Forecasting Game II, a simple auxiliary game between Realty and Forecaster is considered: Forecaster chooses $p_n \in [0,1]$, Realty chooses $\omega_n \in \{0,1\}$. Forecaster losses (and Realty gains) $S(p_n)(\omega_n - p_n)$.

For any mixed strategy of Realty $Q_n \in \mathcal{P}\{0,1\}$, let Forecaster's strategy be $p_n = Q\{1\}$. So, the Realty's expected gain is $S(p_n)(1-Q\{1\})Q\{1\} + S(p_n)(0-Q\{1\})(1-Q\{1\}) = 0$.

In order to apply von Neumann's minimax theorem, which requires that move space be finite, we replace Forecaster move space $[0,1]$ with a finite subset of $[0,1]$ dense enough that the value of the game is smaller than some arbitrary small positive number Δ (depending on n). This is possible, since $|S_n(p)| \leq \mathcal{K}_{n-1} \leq 2^{n-1}$. [2] The minimax theorem asserts that Forecaster has a mixed strategy $P \in \mathcal{P}[0,1]$ such that

$$\int S_n(p)(\omega_n - p)P(dp) \leq \Delta \tag{3}$$

for both $\omega_n = 0$ and $\omega_n = 1$.

Let E_Δ be the subset of $\mathcal{P}[0,1]$ consisting all probability measures P satisfying (3) for $\omega_n = 0$ and $\omega_n = 1$. Endowed with the weak topology, $\mathcal{P}[0,1]$ is compact. Since each E_Δ is closed, $\cap E_{\Delta_i} \neq \emptyset$, where Δ_i, $i = 1, 2, \ldots$, is some decreasing to 0 sequence of real numbers. So there exists $P_n \in \mathcal{P}[0,1]$ such that

$$\int S_n(p)(\omega_n - p)P_n(dp) \leq 0$$

for both $\omega_n = 0$ and $\omega_n = 1$.

In Binary Forecasting Game II, consider the strategy for Forecaster that uses at any round n the probability distribution P_n just defined and uses as his second move the

[2] Skeptic must choose $S_n(p)$ such that $\mathcal{K}_n \geq 0$ for all n no matter the other players move.

function f_n defined $f_n(p) = S_n(p)(\omega_n - p)$. Then $\mathcal{F}_n = \mathcal{K}_n$ for all n. So either Skeptic's capital will stay bounded or Forecaster's capital will be unbounded. $\triangle$

Vovk et al. [11] (see also [10]) also showed that Skeptic can present a strategy $S_n(p)$ such that the winning strategy of Forecaster existing by Theorem 1 announces forecasts $p_1, p_2, \dots$ which are well-calibrated for an arbitrary sequence $\omega_1\omega_2\dots$ of outcomes.

In that follows we consider some version of the Oakes' example in the game-theoretic framework. A different version of this result is given in [12].

We consider some modification of Binary Forecasting Game II in which Skeptic (but not Forecaster) announces $f_n : [0,1] \to \mathcal{R}$. This means that Skeptic defines the test of randomness he needs.

Also, at each step n, Skeptic divide his capital into two accounts: $\mathcal{K}_n = \mathcal{Q}_n + \mathcal{F}_n$; he uses the capital $\mathcal{F}_n$ to force Random Number Generator to generate random numbers which pass the test f_n.

Let $\mathcal{K}_0 = 2$.
FOR $n = 1, 2, \dots$
Skeptic announces $S_n : [0,1] \to \mathcal{R}$.
Forecaster announces a probability distribution $P_n \in \mathcal{P}[0,1]$.
Reality announces $\omega_n \in \{0,1\}$.
Skeptic announces $f_n : [0,1] \to \mathcal{R}$ such that $\int f_n(p)P_n(dp) \le 0$.
Random Number Generator announces $p_n \in [0,1]$.
Skeptic updates his capital $\mathcal{K}_n = \mathcal{K}_{n-1} + S_n(p_n)(\omega_n - p_n) + f_n(p_n)$.
ENDFOR

We divide the Skeptic's capital into two parts:
$\mathcal{K}_n = \mathcal{Q}_n + \mathcal{F}_n$ for all n, where
$\mathcal{Q}_0 = 1$ and $\mathcal{F}_0 = 1$.
$\mathcal{Q}_n = \mathcal{Q}_{n-1} + S_n(p_n)(\omega_n - p_n)$ and
$\mathcal{F}_n = \mathcal{F}_{n-1} + f_n(p_n)$.

Restriction on Skeptic: Skeptic must choose the S_n and f_n so that his capital $\mathcal{K}_n$ is nonnegative for all n no matter how the other players move.

Actually, Skeptic will choose the S_n and f_n so that both of his capitals $\mathcal{Q}_n$ and $\mathcal{F}_n$ are nonnegative for all n no matter how the other players move.

We prove that when Forecaster uses finite subsets of $[0,1]$ for randomization Realty and Skeptic can defeat Forecaster (and Random Number Generator) in this forecasting game, where Realty and Skeptic win if Skeptic's capital $\mathcal{K}_n$ is unbounded; otherwise Forecaster and Random Number Generator win.

Theorem 2 *Assume Forecaster's uses a randomized strategy with a positive level of discreteness on each infinite sequence ω. Then Realty and Skeptic win in the modified Binary Forecasting Game II.*

Proof. Define a strategy for Realty: at any step n Realty announces an outcome

$$\omega_n = \begin{cases} 0 \text{ if } P_n((0.5,1]) > 0.5 \\ 1 \text{ otherwise.} \end{cases}$$

We follow Shafer and Vovk's [9] method of defining the defensive strategy for Skeptic.

Let $\epsilon_k = 2^{-k}$, $k = 1, 2, \dots$. We define recursively by n: $\mathcal{Q}_0^{s,k} = 1$, $S_0^{s,k}(p) = 0$, $s = 1, 2$, and for $n \ge 1$

$$S_n^{1,k}(p) = -\epsilon_k \mathcal{Q}_{n-1}^{1,k} \xi(p > 0.5), \tag{4}$$
$$S_n^{2,k}(p) = \epsilon_k \mathcal{Q}_{n-1}^{2,k} \xi(p \le 0.5), \tag{5}$$

where $\xi(true) = 1$, $\xi(false) = 0$, and for $n \ge 1$

$$\mathcal{Q}_n^{1,k} = \mathcal{Q}_{n-1}^{1,k} + S_n^{1,k}(p_n)(\omega_n - p_n)), \tag{6}$$
$$\mathcal{Q}_n^{2,k} = \mathcal{Q}_{n-1}^{2,k} + S_n^{2,k}(p_n)(\omega_n - p_n)). \tag{7}$$

We combine $S_n^{1,k}(p)$ and $S_n^{2,k}(p)$ in the Skeptic's strategy $S_n(p) = \frac{1}{2}(S_n^1(p) + S_n^2(p))$, where

$$S_n^1(p) = \sum_{k=1}^{\infty} \epsilon_k S_n^{1,k}(p)$$

and

$$S_n^2(p) = \sum_{k=1}^{\infty} \epsilon_k S_n^{2,k}(p).$$

It can be proved by the mathematical induction on n that $0 \le Q_n^{i,k} \le 2^n$ and $|S_n^{i,k}(p)| \le 2^{n-1}$ for $i = 1, 2$ and for all k, p and n. Then these sums are finite for each n and p.

By (6)-(7) the Skeptic's capital $\mathcal{Q}_n$ at step n, when he follows the strategy $S_n(p)$, equals

$$\mathcal{Q}_n = \frac{1}{2} \sum_{k=1}^{\infty} \epsilon_k (\mathcal{Q}_n^{1,k} + \mathcal{Q}_n^{2,k}).$$

Define for each n the function $g_n(p) = (2\xi(p \le 0.5) - 1)(\omega_n - p)$. Let $E_{P_n}(g_n) = \int g_n(p) P_n(dp)$.

Recall that Forecaster uses some randomized strategy P_n, $n = 1, 2, \ldots$.

We define recursively by n: $\mathcal{F}_0^k = 1$, $g_0^k(p) = 0$, and for $n \ge 1$

$$g_n^k(p) = -\epsilon_k \mathcal{F}_{n-1}^k (g_n(p) - E_{P_n}(g_n)), \tag{8}$$

where $\epsilon_k = 2^{-k}$, and for $n \ge 1$

$$\mathcal{F}_n^k = \mathcal{F}_{n-1}^k + g_n^k(p_n) \tag{9}$$

By definition for any k and n,

$$\mathcal{F}_n^k = \prod_{j=1}^{n} (1 - \epsilon_k (g_j(p_j) - E_{P_j}(g_j))). \tag{10}$$

By (10) $0 \le \mathcal{F}_n^k \le 2^n$ for all n and k.

Finally, Skeptic defines at step n

$$f_n(p) = \sum_{k=1}^{\infty} \epsilon_k g_n^k(p).$$

By definition $\int f_n(p) P_n(dp) \le 0$.

By (10) the Skeptic's capital $\mathcal{F}_n$ at step n, when he follows the strategy $f_n(p)$, equals

$$\mathcal{F}_n = \sum_{k=1}^{\infty} \epsilon_k \mathcal{F}_n^k.$$

Also, $\mathcal{F}_n \geq 0$ for all n.

Suppose that $\sup_n \mathcal{F}_n = C < \infty$, where $C > 0$. Then $\sup_n \mathcal{F}_n^k < \frac{C}{\epsilon_k}$ for each k.

We have for each k,

$$\ln \mathcal{F}_n^k \geq -\epsilon_k \sum_{j=1}^n (g_j(p_j) - E_{P_j}(g_j)) - n\epsilon_k^2.$$

Here we use the inequality $\ln(1+r) \geq r - r^2$ for all $|r| \leq \frac{1}{2}$.

Since $\mathcal{F}_n$ is bounded by $C > 0$, we have for any k

$$\frac{1}{n}\sum_{j=1}^n (g_j(p_j) - E_{P_j}(g_j)) \geq \frac{-\ln C + \ln(\epsilon_k)}{n\epsilon_k} - \epsilon_k \geq -2\epsilon_k \tag{11}$$

for all sufficiently large n.

Define two variables

$$\vartheta_{n,1} = \sum_{j=1}^n \xi(p_j > 0.5)(\omega_j - p_j),$$

$$\vartheta_{n,2} = \sum_{j=1}^n \xi(p_j \leq 0.5)(\omega_j - p_j).$$

By definition of g_j,

$$\vartheta_{n,2} - \vartheta_{n,1} = \sum_{j=1}^n g_j(p_j).$$

For technical reason define $g_{1,j}(p) = \xi(p > 0.5)(\omega_j - p)$ and $g_{2,j}(p) = \xi(p \leq 0.5)(\omega_j - p)$. Then $g_j(p) = g_{2,j}(p) - g_{1,j}(p)$.

Assume for any n the probability distribution P_n is concentrated on a finite set $\{p_{n,1}, \dots, p_{n,m_n}\}$.

For technical reason, if necessary, we add 0 and 1 to the support set of P_n and set their probabilities to be 0. Denote $p_n^- = \max\{p_{n,t} : p_{n,t} \leq 0.5\}$ and $p_n^+ = \min\{p_{n,t} : p_{n,t} > 0.5\}$.

By definition ω_n, p_n^+ and p_n^- are predictable and $p_n^+ - p_n^- \geq \Delta$ for all n, where $\Delta > 0$. We have

$$\sum_{j=1}^n E_{P_j}(g_{1,j}) \leq \sum_{\omega_j=0} P_j\{p > 0.5\}(-p_j^+) + \sum_{\omega_j=1} P_j\{p > 0.5\}(1 - p_j^+) \leq$$

$$-0.5\sum_{j=1}^n \xi(\omega_j = 0)p_j^+ + 0.5\sum_{j=1}^n \xi(\omega_j = 1)(1 - p_j^+). \tag{12}$$

$$\sum_{j=1}^n E_{P_j}(g_{2,j}) \geq \sum_{\omega_j=0} P_j\{p \leq 0.5\}(-p_j^-) + \sum_{\omega_j=1} P_j\{p \leq 0.5\}(1 - p_j^-) \geq$$

$$-0.5\sum_{j=1}^n \xi(\omega_j = 0)p_j^- + 0.5\sum_{j=1}^n \xi(\omega_j = 1)(1 - p_j^-). \tag{13}$$

Subtracting (12) from (13), we obtain

$$\sum_{j=1}^{n} E_{P_j}(g_j) = \sum_{j=1}^{n} E_{P_j}(g_{2,j}) - \sum_{j=1}^{n} E_{P_j}(g_{1,j}) \geq 0.5\Delta n.$$

Using (11), we obtain for all sufficiently large n

$$\frac{1}{n}(\vartheta_{n,2} - \vartheta_{n,1}) = \frac{1}{n}\sum_{j=1}^{n} g_j(p_j) \geq \frac{1}{n}\sum_{j=1}^{n} E_{P_j}(g_j) - 2\epsilon_k \geq 0.5\Delta - 2\epsilon_k. \qquad (14)$$

Now we compute a lower bound of Skeptic's capital.

We have from the definition (4)-(5) and (15)-(16).

$$\mathcal{Q}_n^{1,k} = \prod_{j=1}^{n}(1 - \epsilon_k \xi(p_j > 0.5)(\omega_j - p_j)), \qquad (15)$$

$$\mathcal{Q}_n^{2.k} = \prod_{j=1}^{n}(1 + \epsilon_k \xi(p_j \leq 0.5)(\omega_j - p_j)). \qquad (16)$$

By (15) and (16), for $i = 1, 2$, $0 \leq \mathcal{Q}_n^{i,k} \leq 2^n$ for all n no matter how the other players move.

By (15)-(16) at step n

$$\ln \mathcal{Q}_n^{1,k} \geq -\epsilon_k \vartheta_{n,1} - \epsilon_k^2 n, \qquad (17)$$

$$\ln \mathcal{Q}_n^{2,k} \geq \epsilon_k \vartheta_{n,2} - \epsilon_k^2 n. \qquad (18)$$

The inequalities (17), (18) and (14) imply

$$\limsup_{n\to\infty} \frac{\ln \mathcal{Q}_n^1 + \ln \mathcal{Q}_n^2}{n} \geq 0.5\epsilon_k \Delta - 2\epsilon_k^2 \geq 2\epsilon_k^2 \qquad (19)$$

for all sufficiently large n, where $\epsilon_k \leq \frac{1}{8}\Delta$.

From this, we obtain that for $i = 1$ or for $i = 2$,

$$\limsup_{n\to\infty} \frac{\ln \mathcal{Q}_n^{i,k}}{n} \geq \epsilon_k^2$$

for all sufficiently large n.

Hence, we obtain for the total capital of Skeptic $\mathcal{K}^n = \mathcal{Q}^n + \mathcal{F}^n$

$$\limsup_{n\to\infty} \mathcal{K}_n = \infty$$

no matter how Forecaster moves if Realty uses her strategy defined above.

We obtain also a lower bound of calibration error for Binary Forecasting Game II.

Corollary 1 *Assume Forecaster's uses a randomized strategy with a positive level of discreteness on each infinite sequence ω. Then Realty can announce an infinite binary sequence $\omega_1\omega_2\ldots$ such that*

$$\limsup_{n\to\infty} \left|\frac{1}{n}\vartheta_{n,i}\right| \geq 0.25\Delta \qquad (20)$$

for $i = 1$ *or for* $i = 2$.
This inequality immediately follows from (14).

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